

# ON THE STABILITY AND ASYMPTOTIC STABILITY OF THE PERIODIC SOLUTION OF A LIENARD EQUATION

<sup>1</sup>E.O.Eze, <sup>2</sup>U.E. Obasi and <sup>3</sup>S.I Ezeh

<sup>1,2</sup> Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Umuahia. Abia state Nigeria.

Email: [obinwanneeze@gmail.com](mailto:obinwanneeze@gmail.com), [sirurchobasi@gmail.com](mailto:sirurchobasi@gmail.com) and [ijeoma4mary@yahoo.com](mailto:ijeoma4mary@yahoo.com)

Corresponding author: [obinwanneeze@gmail.com](mailto:obinwanneeze@gmail.com), 08033254972

## Abstract

This paper propose a qualitative approach to the periodic solution of a Lienard equation using Lyapunov direct method and Cartwright method to achieve asymptotic stability and hence stability of the solution. Futhermore, Mathcad was applied to demonstrate the numerical behavior of the solution which improves and extends some results in literature.

**Keywords:** Cartwright Method, Asymptotic Stability, Lyapunov Direct Method, Lienard Equation, Stability

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## 1. Introduction

Consider a second order differential equation of the Lienard type

$$\ddot{x} + c\dot{x} + ax = h(t) \tag{1.1}$$

with boundary conditions

$$x(0) = x(2\pi)$$

$$\dot{x}(0) = \dot{x}(2\pi)$$

where  $h(t)$  is one of the following  $T$  –periodic functions:  $h(t) = k\sin(wt)$ ,  $h(t) = k\cos(wt)$  and  $h(t) = e(t)$ .  $\ddot{x}$  is the second derivative with respect to time and  $c, a$  are real constants.

Lienard equation named after French physicist Alfred-Marie Lienard is a second order

differential equation used to model oscillating circuits [1]. The equation is intensely studied during the development of radio and vacuum tube technology. In the presence of a linear restoring and non-linear damping, Lienard equation describes the dynamics of a system with one degree of freedom. These can be seen in the generalized Lienard equation

$$x''(t) + f(x)x' + x = h(t) \quad (1.2)$$

where  $f(x)$  and  $g(x)$  are continuously differentiable on  $\mathbb{R}$ . If the function  $f$  has the property  $f(x) < 0$  for small  $|x|$ ,  $f(x) > 0$  for large  $|x|$  ie. If for small amplitudes the system absorbs energy and for large amplitude dissipation occurs, then in the system one can expect self-existing oscillations [2]. An important special case of the Lienard equation is the Vander Pol equation which is used to model the periodic firing of nerve cells driven by a constant current [3]. Due to importance of Lienard equation in ecological, biological as well as mechanical systems, many researches have used different techniques to obtain solutions of Lienard equation with resounding results. For instance see [4]-[7]. On the asymptotic stability of Lienard equation see [8]-[13] and the reference therein.

This paper is motivated by studying [14] and [15]. The objective of this paper therefore is to investigate the stability and asymptotic stability of the equilibrium point formed by a Lienard equation using Lyapunov direct method. The result presented in this paper is an improvement of the result announced in [16].

This paper is divided into four sections. In section 2, we present some preliminary results and in three, we presented the results and discussion, demonstrating the numerical solution of the Lienard equation using MATHCAD software and in five, we concluded.

## 2. Preliminaries

**Definition 2.1** Consider the system

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \quad (1.3)$$

where  $x: I \rightarrow \mathbb{R}^n$  and  $f: D \subseteq I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  are maps,  $I$  is an interval of real line,  $D$  is an open subset of  $\mathbb{R} \times \mathbb{R}^n$ ;  $n \geq 1$  and  $f$  is such that (1.3) has a unique solution.

(a) Let  $z(t)$  be a solution of (1.3). The solution  $z(t)$  of (1.3) is said to be stable (in the sense of Lyapunov) if given  $\varepsilon > 0$  there exist  $\delta = \delta(t_0, \varepsilon) > 0$  such that all solution  $x(t)$  of (1.3) satisfying  $\|x_0 - z_0\| < \delta$  implies  $\|x(t) - z(t)\| < \varepsilon$  for  $t \geq t_0$

(b) If  $\delta = \delta(\varepsilon)$  only we say that the stability is uniform ie it does not depend on time at any time the solution holds.

(c) The trivial solution  $x = 0$  of (1.3) is said to be asymptotically stable if it is both stable and such that  $\lim_{t \rightarrow \infty} \|x(t)\| = 0$ .

(d) The trivial solution  $x = 0$  of (1.3) is said to be uniformly asymptotically stable if it is both uniformly stable and  $\lim_{t \rightarrow \infty} \|x(t)\| = 0$  holds.

(e) The trivial solution  $x = 0$  of (1.3) is said to be unstable if for any  $\varepsilon > 0$  there exist a  $\delta = \delta(\varepsilon) > 0$  such that all solutions  $x(t)$  of (1.3) satisfying  $\|x_0\| < \delta$  implies  $\|x(t)\| > \varepsilon$  for  $t_1 > t_0$ .

**Remark:** Lyapunov stability means that an arbitrary narrow  $\varepsilon$  –neighbourhood of the solution  $x(t)$  contain all the solutions of (1.3) which sufficiently close to  $z(t_0) = z_0$  at the initial moment  $t_0$ .

**Theorem 2.2** Consider the scalar equation

$$\dot{x} = f(x); \quad x \in \mathbb{R}^n, \quad f(0) = 0 \tag{1.4}$$

where  $f$  is sufficiently smooth. Assume that

(i)  $f \in C^1$

(ii) Then there exist  $C^1$  function  $v: \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $v(x) > 0$  for every  $x$  and  $v(x) = 0$  if  $x = 0$

(iii) Along the solution paths of the equation (1.4)  $\dot{v} \leq 0$ , then the solution  $x = 0$  of equation (1.4) is stable in the sense of Lyapunov.

**Theorem 2.3** Assume that

(i)  $f \in C^1$

(ii) Then there exist  $C^1$  function  $v: \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $v(x) > 0$  for every  $x$  and  $v(x) = 0$  if  $x = 0$

(iii) Along the solution paths of the equation (1.4)  $\dot{v} < 0$ , if  $x \neq 0$  and  $\dot{v} = 0$  ie  $\dot{v}$  is negative definite then the solution  $x = 0$  of equation (1.4) is asymptotically stable in the sense of Lyapunov.

### 3. Results and Discussion

#### 3.1 Stability Analysis by Lyapunov Direct Method.

Lyapunov direct method is an importance concept in stability theory because it defines the behavior of solutions of some nonlinear differential equation and invariably helps to determine the stability of many differential equation [16]. To establish the stability of Lienard equation, it requires that  $h(t) = 0$  which means that

$$\ddot{x} + c\dot{x} + ax = 0 \tag{1.5}$$

Let  $x = x_1, \dot{x}_1 = x_2, \dot{x}_2 = -cx_2 - ax_1$

The equivalent system is given by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -cx_2 - ax_1$$

Let us consider the function  $v: \mathbb{R}^2 \rightarrow \mathbb{R}$  is given by

$$v = \frac{1}{2}x_2^2 + H(x_1) \tag{1.6}$$

where  $H(x_1) = \int_0^{x_1} h(s)ds$

Clearly  $v$  as defined in (1.6) is positive semi-definite. The time derivative  $\dot{v}$  along the solution paths of the equation (1.5) is

$$\dot{v} = x_2\dot{x}_2 + h(x_1)\dot{x}_1 \tag{1.7}$$

where  $h(x_1) = a(x_1)$

$$\dot{v} = x_2(-cx_2 - ax_1) + (ax_1)\dot{x}_1$$

$$\begin{aligned}
 &= -cx_2^2 - ax_1x_2 + ax_1x_2 \\
 &= -cx_2^2
 \end{aligned} \tag{1.8}$$

which is negative definite. Therefore by Lyapunov theorem, the system is asymptotically stable and hence stable in the sense of Lyapunov when  $h(t) = 0$ .

### 3.2 Stability Analysis by Cartwright Method.

Lyapunov functions are vital in determining stability, instability, boundedness and periodicity of ordinary differential. We adopt the method of construction of Lyapunov function used in [17] and extend it to second order differential equation of Lienard type in equation (1.5). The procedure is as follows:

First, we transform equation (1.5) into a system given by

$$\dot{x}_1 = x_2 \tag{1.9}$$

$$\dot{x}_2 = -cx_2 - ax_1 \tag{1.10}$$

Writing compactly, we have

$$\dot{X} = AX \tag{1.11}$$

Where  $A = \begin{bmatrix} 0 & 1 \\ -a & -c \end{bmatrix}$  and  $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  (1.12)

The method discussed here is based on the fact that the matrix  $A$  defined in equation (1.12) has all its eigenvalues with negative real parts. Then from the general theory which corresponds to any positive quadratic form  $U(x)$ , there exists another positive definite quadratic form  $V(x)$  such that

$$V = -U \tag{1.13}$$

We choose the most general quadratic form of order two and pick the coefficient in the quadratic form to satisfy equation (1.13) along the solution paths of equation (1.10). Let  $V$  be defined by

$$2V = Ax_1^2 + Bx_2^2 + 2Kx_1x_2 \tag{1.14}$$

Differentiating equation (1.14) gives

$$\dot{V} = Ax_1\dot{x}_1 + Bx_2\dot{x}_2 + K(\dot{x}_1x_2 + \dot{x}_2x_1) \tag{1.15}$$

$$= Ax_1x_2 + B(-cx_2 - ax_1)x_2 + Kx_2^2 + Kx_1(-cx_2 - ax_1) \tag{1.15}$$

$$= Ax_1x_2 - Bcx_2^2 - Bax_1x_2 + Kx_2^2 - Kcx_1x_2 - Kax_1^2 \tag{1.16}$$

Simplifying the coefficients we have

$$\dot{V} = (A - Ba - Kc)x_1x_2 + (K - Bc)x_2^2 - Kax_1^2 \tag{1.17}$$

To make  $\dot{V}$  negative definite, we adapt the Cartwright method (1956) by equating the coefficient of mixed variable to zero and the coefficients of  $x_1^2$  and  $x_2^2$  to any positive constant (say  $\delta$ ) we have

$$A - Ba - Kc = 0 \tag{1.18}$$

$$K - Bc = \delta \tag{1.19}$$

$$-Ka = \delta \tag{1.20}$$

From equation (1.20)

$$K = -\frac{\delta}{a} \tag{1.21}$$

Then substituting the value of  $K$  into equation (1.19) we obtain

$$B = -\frac{\delta(a+1)}{ca} \tag{1.22}$$

Substituting for  $K$  and  $B$  in (1.18) we have

$$A = -\frac{\delta}{ca} [a + a^2 - c^2] \quad (1.23)$$

The Lyapunov function is gotten by substituting for the values of the constant  $A, B, K$  in equation

(1.14) which gives

$$2V = \frac{\delta}{ca} [x_1^2(a + a^2 - c^2) - x_2^2(1 + a) - 2cx_1x_2] \quad (1.24)$$

$$V = \frac{\delta}{2ca} [x_1^2(a + a^2 - c^2) - x_2^2(1 + a) - 2cx_1x_2]$$

For  $V$  to be positive definite  $\frac{\delta}{ca} = 1$  which gives

$$V(x) = \frac{1}{2} [x_1^2(a + a^2 - c^2) - x_2^2(1 + a) - 2cx_1x_2] > 0$$

Hence at the equilibrium point, the system is asymptotically stable since  $\dot{V} < 0$

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### 3.3. Numerical Solution of Lienard equation

$$a: 0.1 \quad c: 0.1$$

$$t_0 := 0 \quad t_1 := 150 \quad \text{Solution interval endpoints}$$

$$ic := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{Initial condition vector}$$

$$N := 1500 \quad \text{Number of solution values on } [t_0, t_1]$$

$$D(t, X) := \begin{bmatrix} X_1 \\ a \cdot X_0 - c \cdot X_1 \end{bmatrix} \quad \text{Derivative function}$$

$$S := rkfixed(ic, t_0, t_1, N, D)$$

$$T := S^{(0)} \quad \text{Independent variable values}$$

$$X_1 := S^{(1)} \quad \text{Solution function values}$$

$$X_2 := S^{(2)} \quad \text{Derivative function values}$$



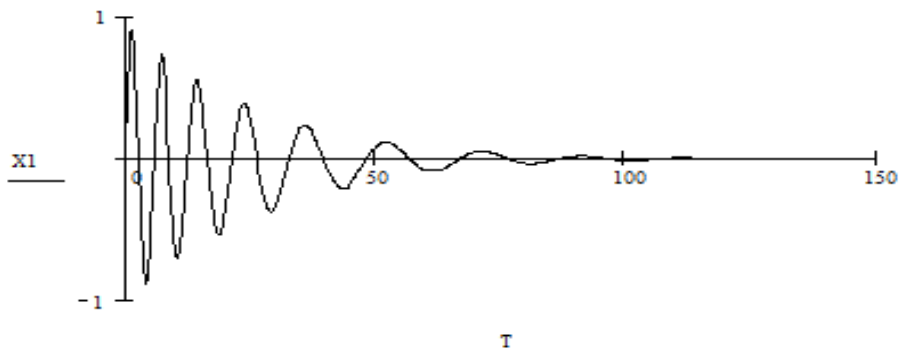


Figure 1: Trajectory profile of Lienard equation

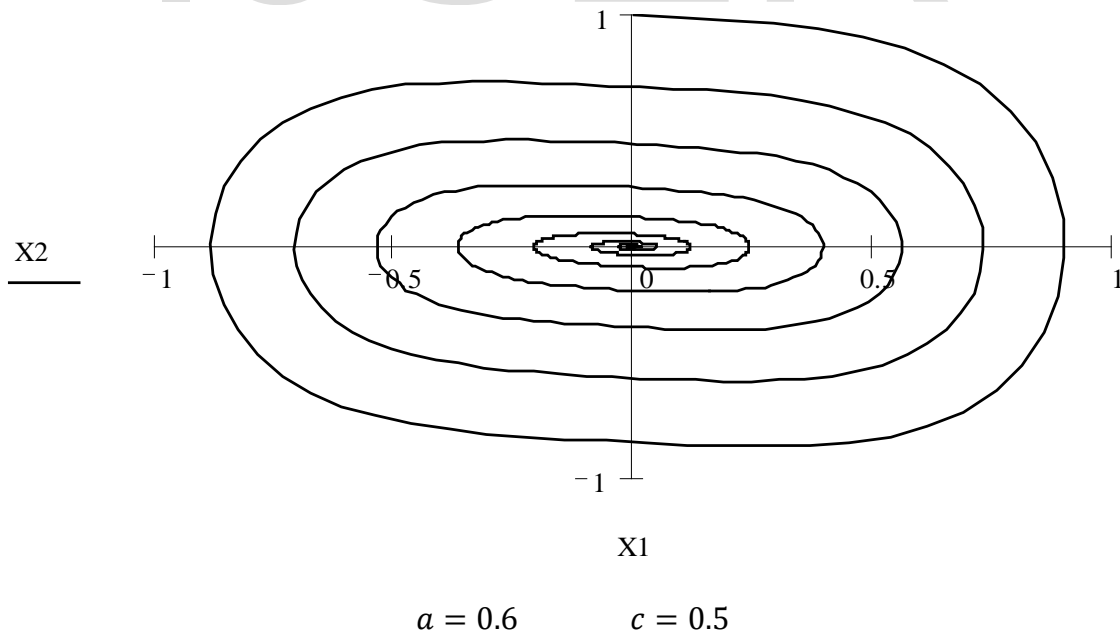


Figure 2: Phase portrait of Lienard equation when  $a = 0.6$  and  $c = 0.5$

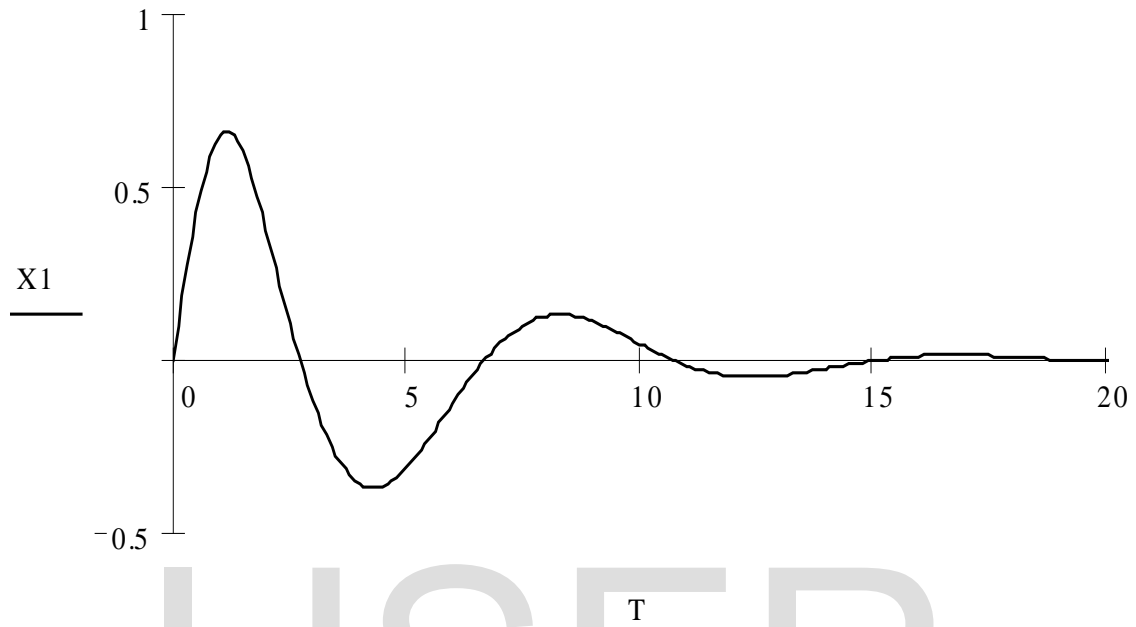


Figure 3: Trajectory profile of Lienard equation

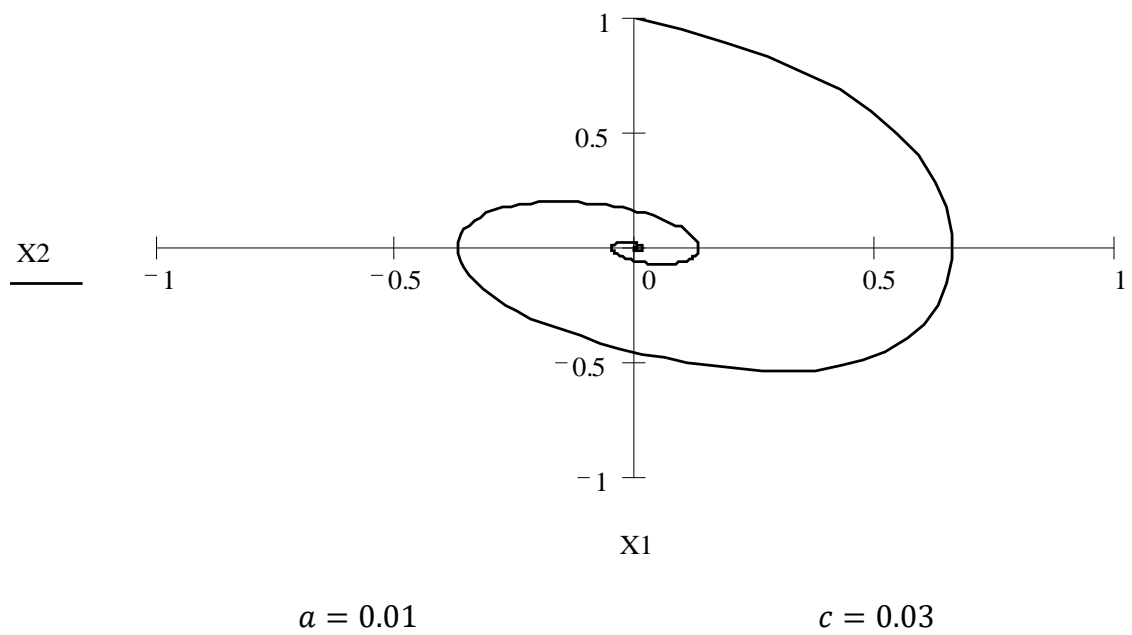


Figure 4: Phase portrait for Lienard equation depicting asymptotic stability of solution as a spiral sink.

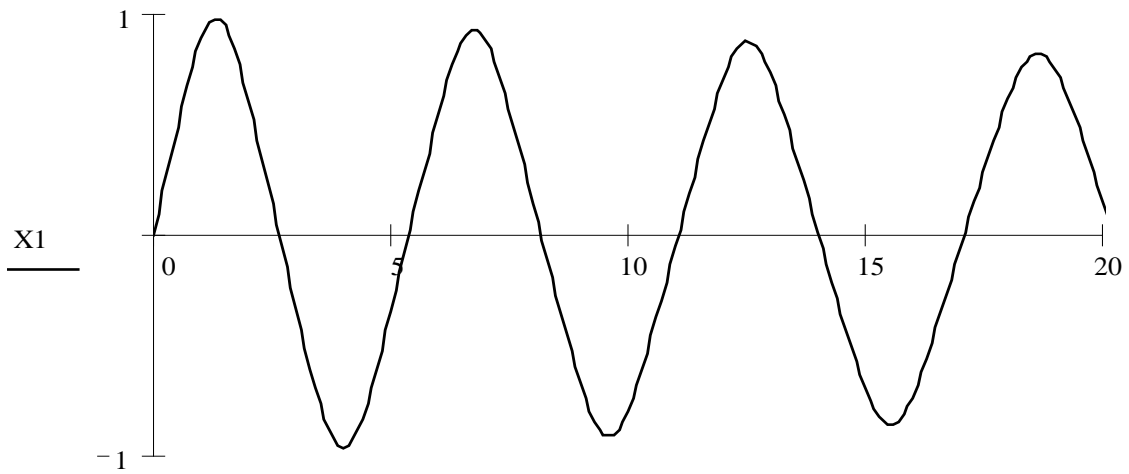


Figure 5: Trajectory profile of Lienard equation

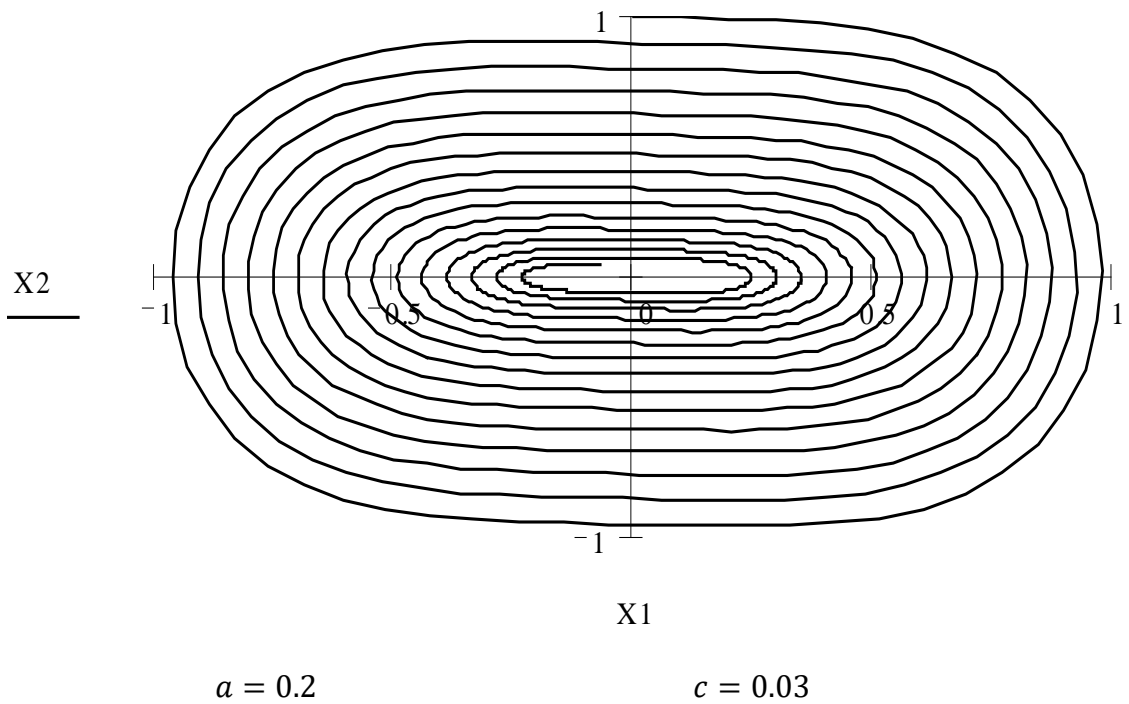


Figure 6: Phase portrait of Lienard equation when  $a = 0.2$  and  $c = 0.03$

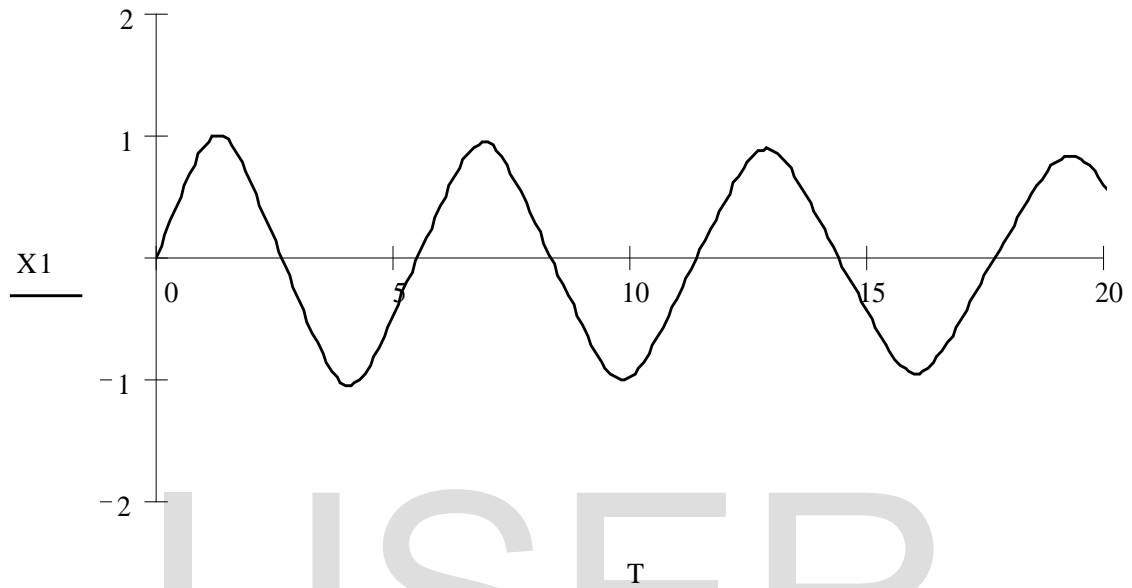
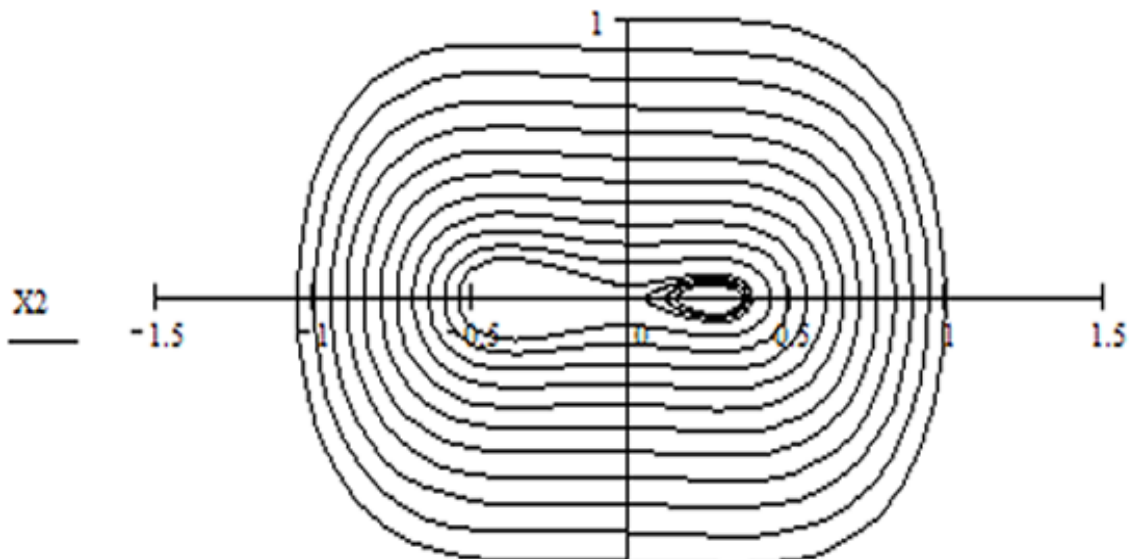


Figure 7: Trajectory profile of Lienard equation



$a = 0.5$

$c = 0.6$

Figure 8: Phase portrait for Lienard equation when  $a = 0.5$  and  $c = 0.6$

#### 4. Conclusion

Lyapunov direct method and Cartwright method has proved to be a useful tool for the analysis of a Lienard type equation. Through some exploits on the first order equivalent systems, we established asymptotic stability and hence the stability of the equation using the two methods. This shows that the equilibrium point of Lienard equation is highly stable. The MATHCAD software has been shown to be effective in supporting the methods used in demonstrating the behavior of the Lienard equation. Using MATHCAD, we obtained the phase portraits and trajectory for different values of  $a$  and  $c$ . The application of this work is found in mechanics where the oscillatory motion of the wheel of a moving vehicle is always directed toward a fixed point. However the disadvantage is that finding a Lyapunov function is more of an art than a science which makes this work open for further research.

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